Lecture 26   5/01/2019

Radiation (Cont'd)

We now consider special cases for a moving point charge and discuss the radiated $\vec{E}$ and $\vec{B}$ fields, and the radiated power, in more detail:

(1) Non-relativistic motion. In this case $\beta \ll 1$ and $1-\vec{\beta}\cdot\hat{n} \simeq 1$. This implies that:

$$\vec{E} \simeq \frac{q}{4\pi\epsilon_0 c} \frac{(\dot{\vec{\beta}}\cdot\hat{n})\hat{n} - \dot{\vec{\beta}}}{R}\Big|_{ret} = \frac{q\,\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{4\pi\epsilon_0 R}\Big|_{ret}$$

$$\vec{B} = \frac{1}{c}\hat{n}\times\vec{E}$$

The instantaneous Poynting vector is:

$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0 c}\vec{E}\times(\hat{n}\times\vec{E}) = \frac{1}{\mu_0 c}|\vec{E}|^2\hat{n}$$

(since $\vec{E}\perp\hat{n}$)

Thus:

$$\frac{dP}{d\Omega} = R^2\vec{S}\cdot\hat{n} = \frac{q^2|\dot{\vec{v}}|^2\sin^2\theta}{(4\pi\epsilon_0)^2\mu_0 c^5}$$

And:

$$P = \int\frac{dP}{d\Omega}d\Omega = \frac{q^2|\dot{\vec{v}}|^2}{(4\pi\epsilon_0)^2\mu_0 c^5}\int\sin^2\theta\, d\Omega \Rightarrow \boxed{P = \frac{q^2|\dot{\vec{v}}|^2}{6\pi\epsilon_0 c^3}}$$

This is the Larmor formula. This formula gives the radiated power in terms of the instantaneous acceleration of the charge at retarded time. For a charge in oscillatory harmonic motion at frequency ω, $\vec{x} = \vec{x}_0 \cos(\omega t + \phi)$, we have $\dot{\vec{v}} = -\omega^2 \vec{x}$ and:

$$P(t') = \frac{q^2 \omega^4 |\vec{x}_0|^2}{6\pi\epsilon_0 c^3} \cos^2(\omega t' + \phi) \Rightarrow \underbrace{\langle P \rangle}_{\text{time average}} = \frac{q^2 |\vec{x}_0|^2 \omega^4}{12\pi\epsilon_0 c^3} = \frac{|\vec{p}|^2 \omega^4}{12\pi\epsilon_0 c^3}$$

This is exactly the same as the expression we saw earlier for the electric dipole emission. Higher-order multipole emission is accounted for through relativistic corrections to the Larmor formula.

(2) Linear motion. In this case, we have $\dot{\vec{\beta}} \parallel \vec{\beta}$. Therefore:

$$\vec{E}(\vec{x}, t) = \frac{q}{4\pi\epsilon_0 c} \left. \frac{\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})}{R(1 - \hat{n} \cdot \vec{\beta})^3} \right|_{ret}$$

$$\vec{B}(\vec{x}, t) = \frac{q}{4\pi\epsilon_0 c^2} \left. \frac{(\dot{\vec{\beta}} \times \hat{n})}{R(1 - \hat{n} \cdot \vec{\beta})^3} \right|_{ret}$$

We see that apart from the relativistic enhancement factor $\frac{1}{(1 - \hat{n} \cdot \vec{\beta})^3}$, there is no difference with the non-relativistic case considered above. This enhancement is angular dependent, and is strogest in the forward direction ($\hat{n}$ parallel to $\vec{\beta}$). The radiated power per unit solid angle is given by:

$$\frac{dP(t)}{d\Omega} = R^2 (\vec{E} \times \vec{H}) \cdot \hat{n} = \frac{q^2}{(4\pi\varepsilon_0)^2 c^3 \mu_0} \frac{[((\dot{\vec{\beta}} \times \hat{n}) \times \hat{n}) \times (\dot{\vec{\beta}} \times \hat{n})] \cdot \hat{n}}{(1 - \hat{n} \cdot \vec{\beta})^6} \Bigg|_{ret}$$

But:

$$((\dot{\vec{\beta}} \times n) \times \hat{n}] \times (\dot{\vec{\beta}} \times \hat{n}) = |\dot{\vec{\beta}} \times \hat{n}|^2 \hat{n} - \cancel{[\hat{n} \cdot (\dot{\vec{\beta}} \times \hat{n})]}^{\,0}\, \hat{n}$$

Hence:

$$\frac{dP(t)}{d\Omega} = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{4\pi c} \frac{|\dot{\vec{\beta}} \times \hat{n}|^2}{(1 - \vec{\beta} \cdot \hat{n})^6} \Bigg|_{ret}$$

Note that:

$$P(t) = \frac{dU}{dt} = \frac{dU}{dt'} \frac{dt'}{dt} = P(t') \frac{dt'}{dt} \qquad (t'\text{: retarded time})$$

As we discussed last time:

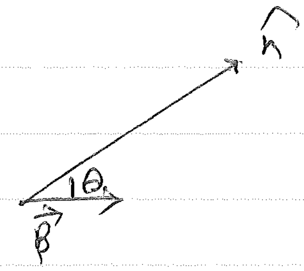
$$\frac{dt}{dt'} = (1 - \vec{\beta} \cdot \hat{n})_{ret}$$

Thus:

$$\frac{dP(t')}{d\Omega} = \frac{dP(t)}{d\Omega} (1 - \vec{\beta} \cdot \hat{n})_{ret} \Rightarrow \boxed{\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi\varepsilon_0} \frac{1}{4\pi c} \frac{|\dot{\vec{\beta}} \times \hat{n}|^2}{(1 - \vec{\beta} \cdot \hat{n})^5}}$$

The angular dependence then follows:

$\hat{n} \cdot \vec{\beta} = |\vec{\beta}| \cos\theta$



And:

$|\dot{\vec{\beta}} \times \hat{n}| = |\dot{\vec{\beta}}| \sin\theta$

linear motion

Resulting in:

$$\frac{dP(t')}{d\Omega} = \frac{q^2}{4\pi\varepsilon_0} \frac{|\dot{\vec{\beta}}|^2}{4\pi c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

In the relativistic limit, $\beta \to 1$, we find:

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \simeq 32\,\gamma^{10} \frac{\sin^2\theta}{\left[\cos\theta + 2\gamma^2(1-\cos\theta)\right]^5} \qquad \left(\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}\right)$$

This is small for $\theta \gg \gamma^{-1}$, and hence the power is highly peaked in the forward direction. To see this, let us use $1-\cos\theta \simeq \frac{\theta^2}{2}$ and $\sin\theta \simeq \theta$.

We then have:

$$\frac{dP(t')}{d\Omega} \simeq \frac{8}{\pi c} \frac{q^2}{4\pi\varepsilon_0} |\dot{\vec{\beta}}|^2 \gamma^{10} \frac{\theta^2}{(1+\gamma^2\theta^2)^5}$$

This implies that the power is beamed along the direction of motion within a narrow cone of half-angle $\theta \sim \gamma^{-1}$.

The total radiated power is:

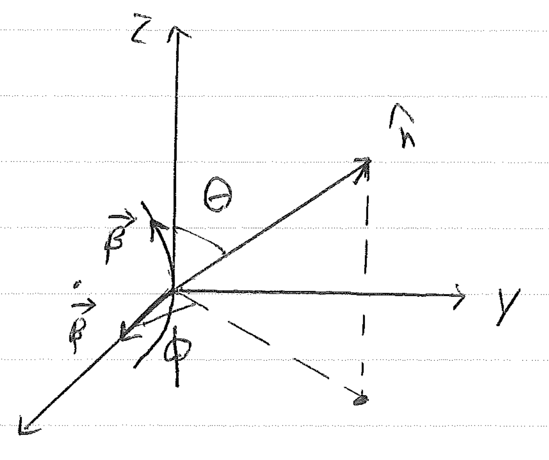
$$P(t') = \int \frac{dP(t')}{d\Omega}\, d\Omega \simeq \frac{8}{\pi c}\, \frac{q^2}{4\pi\varepsilon_0}\, |\dot{\vec{\beta}}|^2 \times 2\pi \gamma^{10} \underbrace{\int_0^\pi \frac{\theta^2}{(1+\gamma^2\theta^2)^5} \sin\theta\, d\theta}_{\gamma^{-2} \int_0^\infty \frac{x}{(1+x)^5}\, dx \quad (x = \gamma^2\theta^2)} \Rightarrow$$

$$\boxed{P(t') = \frac{2}{3c} \left(\frac{q^2}{4\pi\varepsilon_0} \right) |\dot{\vec{\beta}}|^2 \gamma^6}$$

We see the strong enhancement $\propto \gamma^6$ in the relativistic limit.

(3) Circular motion. In this case, we have $\dot{\vec{\beta}} \perp \vec{\beta}$. Let us consider circular motion in the xz plane about a point on the x axis:

We then have:



$$\frac{dP(t')}{d\Omega} = \frac{1}{4\pi c^3} \left(\frac{q^2}{4\pi\varepsilon_0} \right) \frac{|\dot{\vec{v}}|^2}{(1-\beta\cos\theta)^3} \left[1 - \frac{\sin^2\theta \cos^2\phi}{\gamma^2 (1-\beta\cos\theta)} \right]$$

In the relativistic limit, $\gamma \gg 1$, this becomes:

$$\frac{dP(t')}{d\Omega} \simeq \frac{2\gamma^6}{\pi c^3} \left(\frac{q^2}{4\pi\varepsilon_0} \right) |\dot{\vec{v}}|^2 \frac{(1-\gamma^2\theta^2)^2}{(1+\gamma^2\theta^2)^5}$$

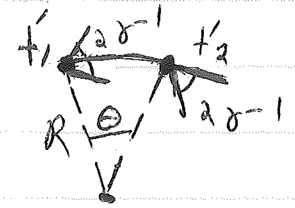
Again, the radiation is highly beamed in the direction of $\vec{\beta}$ ($\theta \sim \gamma^{-1}$).

The total radiated power in this case is:

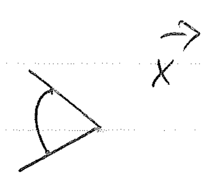
$$P(t') = \int \frac{dP(t')}{d\Omega}\, d\Omega = \boxed{\frac{2}{3c}\left(\frac{q^2}{4\pi\varepsilon_0}\right) |\dot{\vec{\beta}}|\, \gamma^4}$$

This implies that for the same acceleration $|\dot{\vec{\beta}}|$, the power that is radiated in circular motion is a factor γ^2 smaller than that for linear motion.

Circular motion results in "Synchrotron radiation". The spectrum of synchrotron radiation in the relativistic limit can be understood by a heuristic calculation. Consider circular motion on a circle of radius R resulting in radiation observed at a point $\vec{x}$ far away:



The radiation, which is highly beamed, can be observed only if:



$\theta \sim 2\gamma^{-1}$

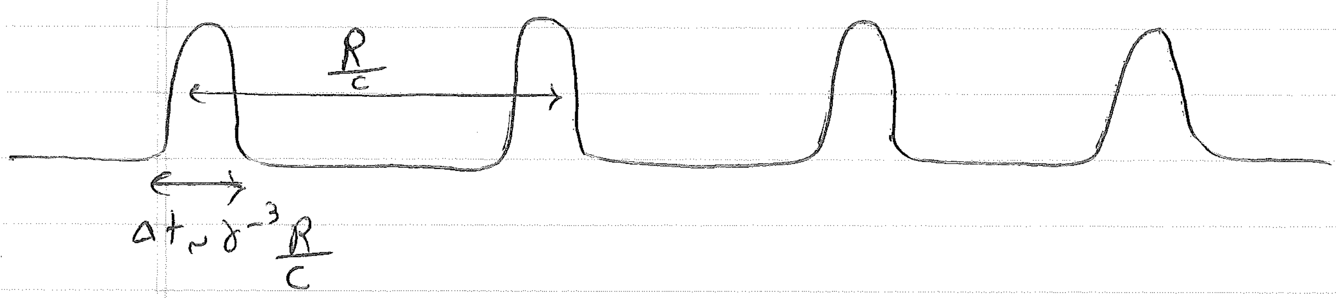
For relativistic circular motion, we have:

$$\Delta t' \equiv t'_2 - t'_1 \sim \frac{R}{\gamma c}$$

The corresponding time interval Δt during which the observer sees radiation is given by (as mentioned before):

$$\Delta t \sim \frac{R}{\gamma c}(1-\beta) \simeq \frac{R}{2\gamma^3 c}$$

Therefore, the observer sees bursts of radiation in pulses of width $\sim \frac{R}{\gamma^3 c}$ separated by intervals $\frac{R}{c}$:



As a result, the spectrum of radiation covers all frequencies up to a maximum frequency $\omega_{max} \sim \gamma^{3}\omega_0$, where $\omega_0 = \frac{2\pi R}{c}$ is the orbital frequency.